Lec. 7

14/03/18

Def. An ordered field $\mathbb{F}$ has the <u>least upper bound property</u> if every nonempty bounded set in $\mathbb{F}$ has a least upper bound in $\mathbb{F}$.

Ex. 1) $\mathbb{R}$ has the l.u.b. property.

2) $\mathbb{Q}$ does <u>not</u>.

Take $\Omega = \{3, 3.1, 3.14, 3.141, 3.1415, 3.14159, 3.141592, \ldots\}$.

In $\mathbb{R}$, $\sup \Omega = \pi$. But Ω has no sup in $\mathbb{Q}$.

Fact. Every ordered field with l.u.b. property is isomorphic to $\mathbb{R}$.

Theorem. Suppose $\Omega \subset \mathbb{R}$ and $\Omega \neq \emptyset$. Then <u>$s = \sup \Omega$</u> if and only if

1) For all $x \in \Omega$, $s \geq x$.

2) For all $\varepsilon > 0$, there exists $x \in \Omega$ s.t. $s - \varepsilon < x$.

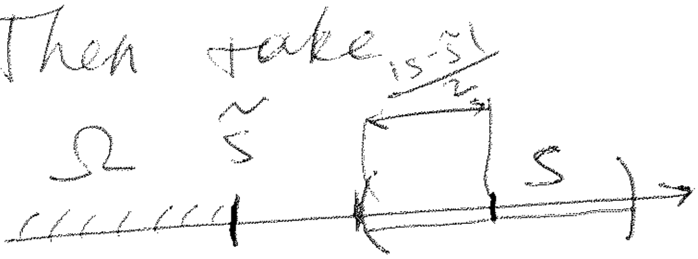
Proof. ($\Rightarrow$) Assume $s = \sup \Omega$. Then

s is an upper bound of $\Omega \Rightarrow$ 1)

Assume 2) is false. For some $\varepsilon > 0$ for every $x \in (s-\varepsilon, s]$, $x \notin \Omega$. This implies $s - \varepsilon$ is an upper bound for Ω. This is a contradiction.

$\Leftarrow$ If 1) holds, then s is an upper bound of Ω. We need to prove any other upper bound $\tilde{s}$ is $\geq s$. Assume $\tilde{s} < s$. Then take

$$\varepsilon = \frac{|\tilde{s} - s|}{2}.$$



Then, obviously,

$x \in (s-\varepsilon, s+\varepsilon) \Rightarrow x > \tilde{s}$.

But then $x \notin \Omega$. So,

$(s-\varepsilon, s+\varepsilon) \cap \Omega = \emptyset$. But this contradicts 2). $\square$

<u>Def.</u> The sequence $(a_n)_{n=1}^{\infty}$ is <u>monotone increasing</u> if $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$.

<u>Note.</u> Sometimes, this is called non-decreasing.

Similarly, define <u>monotone decreasing</u>:

If it's either mon. incr. or mon. decr., it's a <u>monotone sequence</u>.

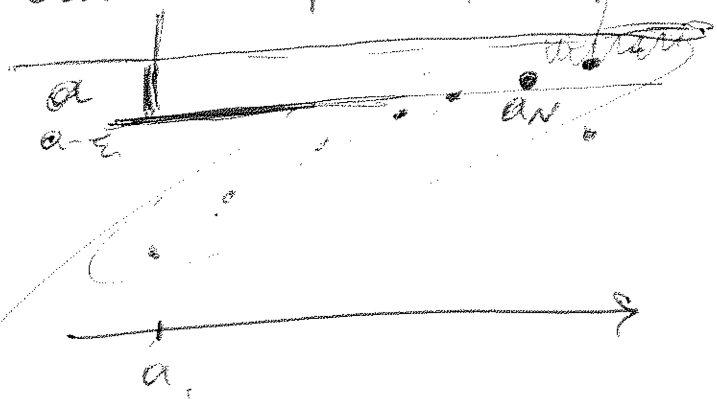
<u>Theorem.</u> A monotone seq. converges if and only if it is bounded.

<u>Proof.</u> ($\Rightarrow$) Did before.

($\Leftarrow$) Assume a_n is mon. incr.

Prove that $\lim_{n\to\infty} a_n = \sup\{a_1, a_2, a_3, \dots\}$.

Call this sup a.

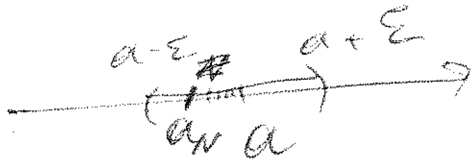


To show $a = \lim_{n\to\infty} a_n$, take $\varepsilon > 0$.

Because $a = \sup\{a_1, a_2, \dots\}$, there exists $N \in \mathbb{N}$ s.t. $a_N \in (a - \varepsilon, a]$.

Then, for all $n \geq N$, we have

$a_n \geq a_N > a - \varepsilon$ and



$a_n \leq a$.

This means $|a - a_n| < \varepsilon$ for such n. $\square$

Lemma. Bernoulli inequality.

1) Given $x \geq -1$ an $n \in \mathbb{N} \cup \{0\}$, we have

$$(1+x)^n \geq 1+nx$$

(We're assuming $x=-1$ and $n=0$ do not hold at the same time).

2) We have $>$ in the above ~~formula~~ if and only if $n>1$ and $x \neq 0$.

Proof - later.

Theorem.

Define $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ by

$$a_n = \left(1+\frac{1}{n}\right)^n \text{ and } b_n = \left(1+\frac{1}{n}\right)^{n+1}.$$

Then a_n is mon. incr., b_n mon. decr.

and $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n$.

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Proof. Let's prove monotonicity.

Note that $a_n = \left(\frac{n+1}{n}\right)^n$, $b_{n-1} = \left(\frac{n}{n-1}\right)^n$.

Now, if $n>1$, we have

$$\frac{a_n}{b_{n-1}} = \left(\frac{(n-1)(n+1)}{n^2}\right)^n = \left(\frac{n^2-1}{n^2}\right)^n$$

$$= \left(1-\frac{1}{n^2}\right)^n = \left(1+\left(-\frac{1}{n^2}\right)\right)^n$$

Use Bernoulli with $x = -\frac{1}{n^2}$.

$$> 1+n\left(-\frac{1}{n^2}\right) = 1-\frac{1}{n} = \underline{\left(\frac{n}{n-1}\right)^{-1}}.$$

Now, $$\frac{b_{n-1}}{a_n} = \left(\frac{n}{n-1}\cdot\frac{n}{n+1}\right)^n$$

$$= \left(\frac{n^2}{n^2-1}\right) = \left(\frac{n^2-1+1}{n^2-1}\right)^n = \left(1+\frac{1}{n^2-1}\right)^n$$

$$> \left(1+\frac{1}{n^2}\right)^n \overset{\text{Bernoulli}}{>} 1+n\cdot\frac{1}{n^2} = \underline{1+\frac{1}{n}}.$$

Thus, $\frac{a_n}{b_{n-1}} > \left(\frac{n}{n-1}\right)^{-1}$ and

$\frac{b_{n-1}}{a_n} > 1+\frac{1}{n}$. This implies

$$\underline{a_n} > \left(\frac{n}{n-1}\right)^{-1} b_{n-1} = \left(\frac{n}{n-1}\right)^{-1}\left(\frac{n}{n-1}\right)^n$$

$$= \left(\frac{n}{n-1}\right)^{n-1} = \left(\frac{n-1+1}{n-1}\right)^{n-1}$$

$$= \left(1+\frac{1}{n-1}\right)^{n-1} = \underline{a_{n-1}}.$$

Similarly, $\underline{b_{n-1}} > \left(1+\frac{1}{n}\right)a_n$

$$= \left(1+\frac{1}{n}\right)\left(1+\frac{1}{n}\right)^n = \left(1+\frac{1}{n}\right)^{n+1} = b_n.$$

This yield the monotonicity of $(a_n)_{n=1}^{\infty}$, $(b_n)_{n=1}^{\infty}$.

To show these seqs. are bounded, observe that

$$2 \overset{=}{} a_1 < \underbrace{a_n < b_n}_{< e <} < b_1 = 4.$$

Because these seqs. are monotone and bounded, they converge. Moreover,

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n}\right)^n$$

$$= \lim_{n\to\infty}\left(1+\frac{1}{n}\right) \cdot \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

$$= 1 \cdot \lim_{n\to\infty} a_n.$$

$\square$

Thus, $\lim_{n\to\infty} b_n = \lim_{n\to\infty} a_n$.

We call this common limit e.

Taking $n=5$, we find $e \in [2,3]$.

$e = 2.7$ [Tolstoy's birth year] . . .

Proof of Bernoulli.

~~Notice~~ Notice that it always holds for $x=-1$.

$$(1+x)^n \geq 1+nx. \qquad (*)$$

Assume $x \neq -1$. $(*)$ holds for $n=0$. Assume it holds. Need to prove

$$(1+x)^{n+1} \geq 1+(n+1)x.$$

$$(1+x)^{n+1} = (1+x)(1+x)^n \geq (1+x)(1+nx)$$

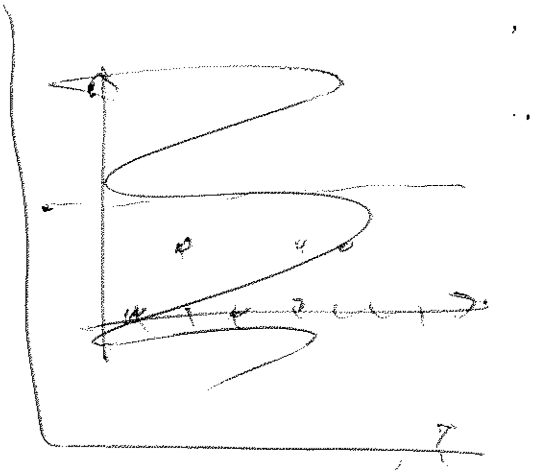
$$= 1+nx+x+nx^2 = \underline{1+(n+1)x} + \underbrace{nx^2}_{\geq 0}$$

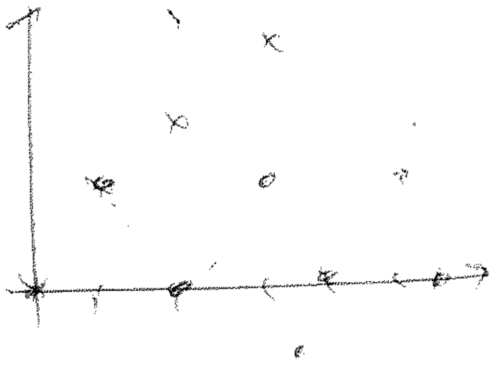
$$\geq 1+(n+1)x.$$

To show the "$>$" part holds, repeat the argument with obvious modifications. $\square$

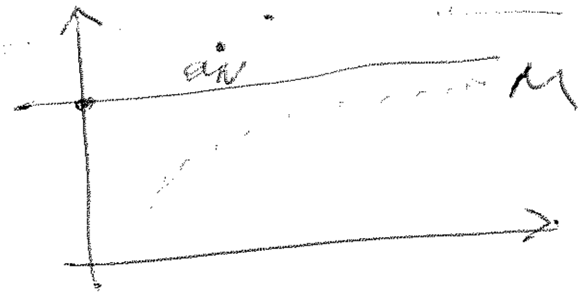
Notation.

We write $\lim_{n\to\infty} a_n = \infty$ if for every $M \in \mathbb{R}$ there exist $N \in \mathbb{N}$ s.t. $n \geq N \Rightarrow a_n > M$.





Similarly, we can define $\lim_{n\to\infty} a_n = -\infty$.



Examples. $\lim_{n\to\infty} n = \infty$,

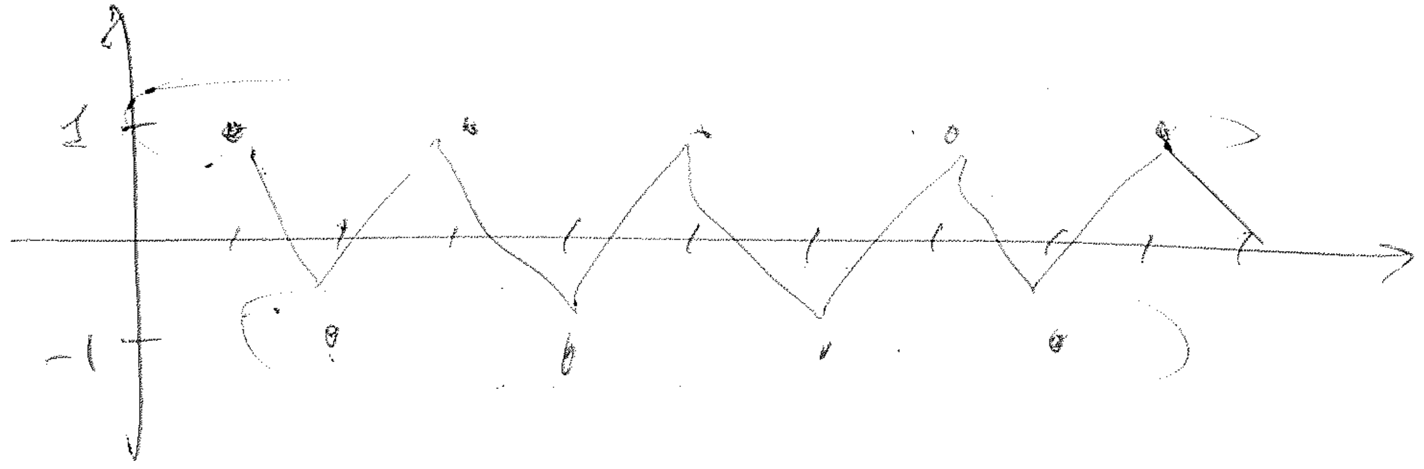
$\lim_{n\to\infty} -n = -\infty$.

Def. Given $(a_n)_{n=1}^{\infty} \subset \mathbb{R}$, consider

$$x_n = \sup\{a_n, a_{n+1}, a_{n+2}, \dots\},$$

$$y_n = \inf\{a_n, a_{n+1}, a_{n+2}, \dots\}.$$

The number $\lim_{n\to\infty} x_n$ is called the $\limsup_{n\to\infty} a_n$ (upper limit); the number $\lim_{n\to\infty} y_n$ is the $\liminf_{n\to\infty} a_n$ (lower limit).



Example. Take this seq.

$\{0, 1, -1, 0, 1, -1, 0, 1, -1, \dots\} = (a_n)_{n=1}^{\infty}$

Then $x_n = 1$ for all n,

$y_n = -1$

Thus, $\limsup_{n\to\infty} a_n = 1$, $\liminf_{n\to\infty} a_n = -1$.

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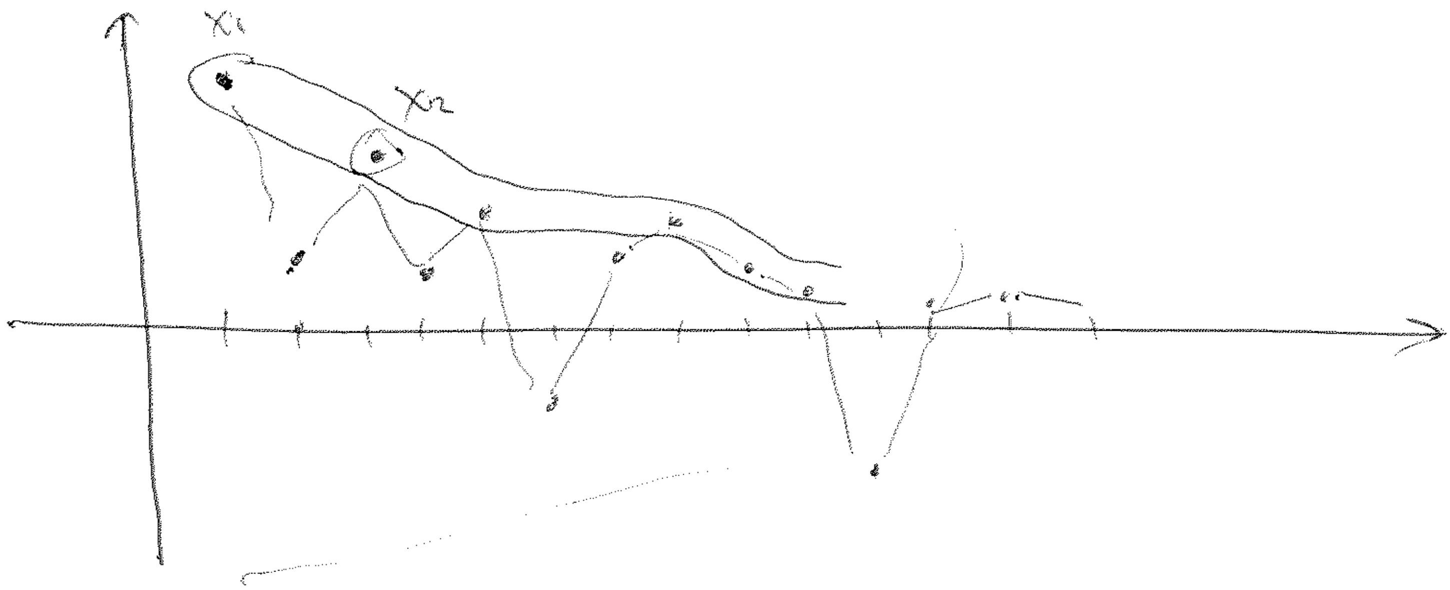
Consider

$x_n = \sup\{a_n, a_{n+1}, a_{n+2}, \dots\}$,

$y_n = \inf\{a_n, a_{n+1}, a_{n+2}, \dots\}$.



$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} x_n,$$

$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty} y_n.$$

Examples. 1) $(a_n)_{n=1}^{\infty} = \{0, 1, -1, 0, 1, -1, \dots\}$

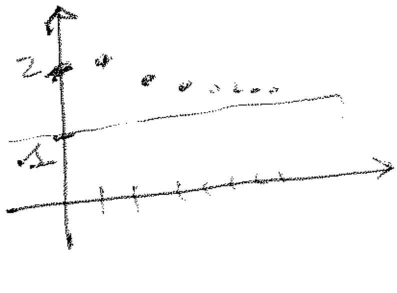
$x_1 = 1,\ x_2 = 1,\ x_3 = 1, \dots$

$\limsup_{n\to\infty} a_n = 1,\quad \liminf_{n\to\infty} a_n = -1.$

2) $\limsup_{n\to\infty} n = \infty.$

$\liminf_{n\to\infty} n = \lim_{n\to\infty} n = \infty.$

3) $\limsup_{n\to\infty} \left(1 + \frac{1}{n}\right) = 1.$



$x_n = 1 + \frac{1}{n},\quad \lim_{n\to\infty} x_n = 1.$

$y_n = 1,\quad \lim_{n\to\infty} y_n = \liminf_{n\to\infty} \left(1 + \frac{1}{n}\right) = 1.$